Brown-York Energy and Radial Geodesics

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We compare the Brown-York (BY) and the standard Misner-Sharp (MS) quasilocal energies for round spheres in spherically symmetric space-times from the point of view of radial geodesics. In particular, we show that the relation between the BY and MS energies is precisely analogous to that between the (relativistic) energy E of a geodesic and the effective (Newtonian) energy E_{eff} appearing in the geodesic equation, thus shedding some light on the relation between the two. Moreover, for Schwarzschild-like metrics we establish a general relationship between the BY energy and the geodesic effective potential which explains and generalises the recently observed connection between negative BY energy and the repulsive behaviour of geodesics in the Reissner-Nordstrøm metric. We also comment on the extension of this connection between geodesics and the quasilocal BY energy to regions inside a horizon.

1 Introduction

It is a consequence of the fundamental general covariance of general relativity that there is no well-defined covariant notion of the local energy density of the gravitational field. The next best thing is perhaps the notion of a quasilocal energy (QLE), i.e. the energy contained in a two-dimensional surface. Numerous definitions of QLE have been proposed in the literature (for a detailed and up-to-date review with many references see [1]), and these tend to be mutually inequivalent even in simple cases such as the Kerr metric [2].

There is at least one case, however, in which there appears to be *almost* universal agreement as to what the QLE should be, namely for round spheres (i.e. orbits of the rotational isometry group) in spherically symmetric space-times. In that case, the classical Misner-Sharp (MS) energy [3] (see e.g. [1] or [4] for recent discussions) is widely considered to be the "standard" definition of the energy for round spheres.

One serious contender to this definition is based on the Brown-York (BY) QLE [5]. The definition of the BY energy is based on the covariant Hamilton-Jacobi formulation of general relativity, and this makes it a natural object to consider in a variety of contexts, with numerous attractive features. However, the standard BY energy for round spheres does not agree with the standard MS energy (even for the Schwarzschild metric), and this fact has occasionally been used as an argument against the BY energy as a "good" definition of a QLE (see e.g. the discussions in [1, 4]).

In this article we will look at the relationship and differences between the MS and BY energies for round spheres from the point of view of geodesics and their associated energy concepts like the relativistic geodesic energy and the effective Newtonian potential. In general, one would not expect point-like objects to be able to probe something not quite local like a QLE. However, the situation is different for round spheres for which the QLE is independent of the angular coordinates. In such a situation it is fair to ask whether there is a relation between the gravitational energy as felt by a point-like observer (geodesic) and that defined according to some QLE prescription.

Originally, our investigation of these issues was prompted by an observation and a remark in [6]. There it was observed that for the Reissner-Nordstrøm metric the BY energy becomes negative for sufficiently small radius. In [6] it was suggested that this negative energy is strictly related to the well-known repulsive behaviour exhibited by the geodesics of massive neutral particles in the Reissner-Nordstrøm metric.

What supports this point of view is the fact that the energy indeed becomes negative at precisely the radius where radial geodesics begin to experience the repulsive behaviour of the Reissner-Nordstrøm core. This clearly hints at a deeper connection between geodesic and quasilocal energy, or, in the words of [6]:"*The turnaround radius agrees with the radius where the quasilocal energy becomes negative, so it seems that the two effects are very likely connected.*" We will indeed be able to establish a general relationship between the BY energy and the geodesic effective potential (for radial geodesics) and, in particular, a relation between negative BY energy and a repulsive behaviour for geodesics.

Our results also shed some light on the difference between the MS and BY energies for round spheres. In particular, for Schwarzschild-like metrics $ds^2 = -f(r)^2dt^2 + f(r)^{-2}dr^2 + r^2d\Omega^2$, for which geodesics are conveniently described in terms of an effective Newtonian potential, we observe that the MS energy is directly related to the effective potential $V_{\text{eff}}(r)$ for radial goedesics, and that the relation between the MS and BY energies is strictly analogous to the relation $E_{\text{eff}} = \frac{1}{2}(E^2 - 1)$ between the energy appearing in the effective potential equation and the relativistic geodesic energy E of the particle. Therefore, inasmuch as E is a relativistic energy and E_{eff} an effective Newtonian energy, perhaps one interpretation of the difference between the BY and MS energies for round spheres is to say that the former provides one with a relativistic notion of gravitational energy while the MS energy is more like an effective Newtonian quantity.

We also briefly discuss the extension of these results to regions inside a horizon. An extension of the BY energy to this case was proposed in [6]. However, it has been remarked[1] that the proposal of [6] should perhaps better be thought of as a quasi-local momentum. Our geodesic perspective is compatible with this point of view since, as we will show, the relation between the BY "energy" of [6] and the effective potential inside the horizon is identical to that between E and E_{eff} provided that one considers geodesics that are *spacelike* (outside the horizon).

We believe that the message of this work is two-fold: First of all, it shows that there are situations where geodesic test particles can be useful to probe candidate definitions of QLE. Moreover, these results also illuminate the difference between the BY and MS energies and provide further evidence that the BY definition of a QLE provides a good (relativistic) measure of the gravitational energy even though (or even precisely because) it does not agree with the standard (and perhaps somewhat more Newtonian) MS energy for round spheres.

2 Brown-York and Misner-Sharp Energy for Spherical Symmetry

We briefly recall the definition of the BY and MS energies for round spheres, referring to the original literature (e.g. [5, 7, 6] and [3]) and the review article [1] for details. We will consider a general spherically symmetric metric written in the form

$$ds^2 = -N(t,r)^2dt^2 + f(t,r)^{-2}dr^2 + r^2d\Omega^2 \tag{1}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ the standard line-element on the unit 2-sphere. Even though we will only consider 4-dimensional space-times in this article, the extension to higher dimensions is rather

[1]e.g. by one of the referees, and by Ruth Durrer (private communication)

straightforward. In contrast to [7] we prefer to work directly with the area radius r as the radial coordinate. The *round spheres* in this space-time (the orbits of the rotational isometry group) are the 2-spheres $t = \text{const.}, r = \text{const.}$ It was shown in [5, 7] that, in any region in which ∂_t is timelike and ∂_r is spacelike, the standard BY quasilocal energy $E_{BY}(t,r)$ associated to a round sphere of radius r, and calculated with respect to the standard static observers associated to the spatial slicing $t = \text{const.}$ is given by

$$E_{BY}(t,r) = \frac{r}{G_N}(1 - f(t,r)) \ , \tag{2}$$

where G_N is Newton's constant.

This BY energy differs from the "standard" Misner-Sharp (MS) energy [3] for round spheres which, for a metric of the type (1) and for any (t,r), is given by

$$E_{MS}(t,r) = \frac{r}{2G_N}\left(1 - f(t,r)^2\right) \ . \tag{3}$$

For example, for the Reissner-Nordstrøm metric $N(r)^2 = f(r)^2 = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$ one has

$$\begin{aligned} E_{BY}(r) &= \frac{r}{G_N}\left(1 - \sqrt{1 - \frac{2m}{r} + \frac{e^2}{r^2}}\right) \\ E_{MS}(r) &= \frac{1}{G_N}\left(m - \frac{e^2}{2r}\right) \ . \end{aligned} \tag{4}$$

Both reduce to the ADM mass $M = m/G_N$ asymptotically, $\lim_{r\to\infty} E_{BY}(r) = \lim_{r\to\infty} E_{MS}(r) = M$ (and for the Schwarzschild metric one evidently has $E_{MS}(r) = M$ for all r). Moreover, for sufficiently small values of r, $r < r_0 = e^2/2m$ both the MS and the BY energy are negative (note that the expression for the BY energy is also valid inside the inner horizon r_- and that $r_0 < r_-$). The qualitative (and not just quantitative) difference bewteen the BY and MS energies is e.g. illustrated by the fact that, unlike the MS energy, the BY energy is finite at $r = 0$, $E_{BY}(0) = -|e|/G_N$ [6].

3 Brown-York and geodesic energy for Schwarzschild-like metrics

In order to analyse the BY energy (and its relationship with the MS energy) from the point of view of geodesics, we now specialise to Schwarzschild-like metrics, i.e. static spherically symmetric metrics with $N(r) = f(r)$ (the extension to time-dependent Schwarzschild-like metrics with $f = f(t,r)$ is straightforward),

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\Omega^2 \ . \tag{5}$$

In this case the behaviour of timelike radial geodesics is governed by the effective potential equation

$$\tfrac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = E_{\text{eff}} \ , \tag{6}$$

where the effective (and effectively Newtonian) potential $V_{\text{eff}}(r)$ is related to $f(r)^2$ by

$$f(r)^2 = 1 + 2V_{\text{eff}}(r) \ , \tag{7}$$

and the effective energy E_{eff} is given in terms of the relativistic geodesic energy per unit rest mass $E = f(r)^2 \dot{t}$ of the particle by

$$E_{\text{eff}} = \tfrac{1}{2}(E^2 - 1) \ . \tag{8}$$

For later we note that $E = f(r_m)$ where r_m (the index could indicate a minimum or maximum) is a turning point, $\dot{r}_m = 0$, of the trajectory, and that in the asymptotically flat case (which we take

here to simply mean $\lim_{r\to\infty} f(r) = 1$) for scattering trajectories that reach (or start out at) $r \to \infty$ one also has the relation $E^2 = 1 + \dot{r}_\infty^2 \geq 1$ between E and the velocity at infinity. In particular, for scattering trajectories in the Reissner-Nordstrøm metric, for the minimal radius $r_m = r_m(E)$ one has $r_m(E) \leq r_m(E = 1) = e^2/2m$, which, as noted in [6], agrees with the radius r_0 where the BY (and MS) energy becomes negative.

In order to now study the relations among the Brown-York energy, the Misner-Sharp energy, and the geodesic effective potential, it turns out to be convenient to introduce the corresponding potentials

$$V_{BY}(r) := -G_N \frac{E_{BY}(r)}{r} \qquad\qquad V_{MS}(r) := -G_N \frac{E_{MS}(r)}{r} \quad . \tag{9}$$

Using the definition (3) of the MS energy and (7), one immediately sees that

$$V_{MS}(r) = -\tfrac{1}{2}(1 - f(r)^2) = V_{\text{eff}}(r) \quad . \tag{10}$$

Thus the MS potential agrees on the nose with the effective potential and has a clear physical interpretation in the present context. In particular, negative MS energy is strictly correlated with a repulsive behaviour of the effective potential for radial geodesics.

What about the BY potential? Given the above relation between the MS energy and radial geodesics, one's first thought may perhaps be[2] that to establish a link with the BY energy one should calculate the latter for freely falling (geodesic) rather than static observers, or for static observers in comoving (Novikov) coordinates. The Schwarzschild BY energy for geodesic obervers was first determined in [8] and more recently, also motivated by the appearance of the first version of the present article on the arXiv, in [9] (with a slightly different prescription). For example, the result of [8] (for an observer initially at rest at infinity) is

$$E_{BY}^{\text{freefall}}(r) = \frac{r}{G_N}\left(\sqrt{1 + \frac{2m}{r}} - 1\right) \quad . \tag{11}$$

In Novikov coordinates (τ, R), where τ is the proper time of a radially infalling observer and R is related to the maximal radius r_m of the geodesic by $R = \sqrt{\frac{r_m}{2m} - 1}$, the Schwarzschild metric reads (see e.g. [10, §31.4])

$$ds^2 = -d\tau^2 + \frac{R^2 + 1}{R^2}\left(\frac{\partial r}{\partial R}\right)^2 dR^2 + r(\tau, R)^2 d\Omega^2 \quad . \tag{12}$$

Calculating the BY energy for "static" observers in this space-time, one finds

$$E_{BY}^{\text{Novikov}}(r) = \frac{r}{G_N}\left(1 - \frac{R}{\sqrt{R^2+1}}\right) = \frac{r}{G_N}\left(1 - \sqrt{1 - \frac{2m}{r_m}}\right) \quad . \tag{13}$$

This can e.g. be seen to agree with the result of [9], based on the calculation of the freefall BY energy in Kruskal coordinates. Thus, neither do the above results reproduce the MS energy, nor do they appear to be related to the effective geodesic potential in any other particularly useful or illuminating way.

In this context it is perhaps also worth pointing out that in [7] a change of coordinates (foliation) $t \to T(t, r)$ for the Schwarzschild metric was exhibited with respect to which the standard BY energy takes the MS value $E_{BY}(r) = M$. Such a foliation, giving $E_{BY}(r) = E_{MS}(r)$, can also

[2]This was not our first thought, but we are grateful to one of the referees for reminding us that it should perhaps have been.

readily be constructed for the general Schwarzschild-like metric (5). It suffices to choose $T(t,r)$ such that

$$dT = dt + \frac{1-f^2}{f^2(1+f^2)}dr \ . \tag{14}$$

To see this note that for a general spherically symmetric metric of the form

$$ds^2 = -N(t,r)^2 dt^2 + F(t,r)^{-2}(dr + A(t,r)dt)^2 + r^2 d\Omega^2 \tag{15}$$

the BY energy is still given by (2) (with $f \to F$), and that with the choice (14) one has $1 - F = \frac{1}{2}(1-f^2)$, so that indeed $E_{BY}(r) = E_{MS}(r)$. However, the physical significance of this choice of foliation escapes us, and this construction does not appear to shed any light on the relationship between the BY energy and geodesic notions of energy.

Thus we now return to the task of relating the standard BY energy (2) to the geodesic effective potential. Substituting (7) in (2), one finds

$$E_{BY}(r) = \frac{r}{G_N}(1 - \sqrt{1 + 2V_{\text{eff}}(r)}) \ , \tag{16}$$

or

$$1 + V_{BY}(r) = \sqrt{1 + 2V_{\text{eff}}(r)} \ . \tag{17}$$

While this relation, which we may also read as the relation between the MS energy and the BY energy, may appear to be somewhat obscure, it reveals several interesting features of the BY potential $V_{BY}(r)$ and its relation to $V_{\text{eff}}(r) = V_{MS}(r)$:

1. First of all we observe that (17) and (6) allow us to express the BY potential in terms of geodesic quantities as

   $$1 + V_{BY}(r) = \sqrt{E^2 - \dot{r}^2} \leq E \ . \tag{18}$$

   In other words, the relation between the Brown-York potential and the relativistic energy E (per unit rest mass) of the particle can be phrased as

   The energy E of the geodesic particle is greater or equal to the sum of its rest mass and the gravitational potential energy (as measured by $V_{BY}(r)$), with equality at points where $\dot{r} = 0$.

   Thus the BY potential appears to provides a reasonable measure of the energy of the gravitational field in this context. The inequality $E \geq 1 + V_{BY}(r)$ should be compared and contrasted with the analogous equation $E_{\text{eff}} \geq V_{MS}(r)$ for the MS (or effective) potential that follows from (6). This suggests a certain analogy $E_{BY} \leftrightarrow E$ and $E_{MS} \leftrightarrow E_{\text{eff}}$.

2. This analogy is strengthened by the observation that (17) implies

   $$V_{\text{eff}}(r) = \tfrac{1}{2}\left((1 + V_{BY}(r))^2 - 1\right), \tag{19}$$

   which shows that the relation between V_{eff} and $1 + V_{BY}$ is identical to the relation $E_{\text{eff}} = \frac{1}{2}(E^2 - 1)$ (8) between the effective energy E_{eff} and the geodesic particle energy E.

   Thus, since E is a relativistic energy and E_{eff} an effective Newtonian quantity, it is tempting to say that the BY energy provides one with a relativistic notion of gravitational energy while the MS energy is really more like an effective Newtonian quantity. So far, however, this is only a suggestion, based on the geodesic analogy that we have developed here, and further analysis of this issue, in other settings, will be required to substantiate (or disprove) this interpretation of the difference between E_{MS} and E_{BY}.

3. Finally, (17) implies that $V_{\text{eff}}(r)$ and $V_{BY}(r)$ have the same zeros and that the BY potential is repulsive/positive whenever (and whereever) the effective potential is repulsive. Thus the BY energy is negative if and only if the effective potential is repulsive. In particular, (18) leads to a simple expression for the BY energy at any turning point r_m ($\dot{r}_m = 0$) of the potential, namely $1 + V_{BY}(r_m) = E$ or
$$E_{BY}(r_m) = \frac{r_m}{G_N}(1 - E) \ . \tag{20}$$
This also follows directly from the definition (2) and the previously noted $E = f(r_m)$. In particular, $E_{BY}(r_m)$ is negative for scattering trajectories with $E > 1$. Thus non-positive BY energy is necessary for a repulsive behaviour of radial geodesics. This provides a simple explanation and proof of a generalisation of the observation made in [6] in the context of the Reissner-Nordstrøm metric. Note also that, for the Schwarzschild metric, at $r = r_m$ one has $E_{BY}(r_m) = E_{BY}^{\text{Novikov}}(r_m)$, so that static and freely falling observers can agree on the energy at a turning point of the freely falling observer, as they should.

All in all this provides us with a coherent picture of the relation between geodesic notions of energy on the one hand, and the quasilocal gravitational MS and BY energies for round spheres on the other.

Finally we comment briefly, from the present geodesic point of view, on the extension $E_{LSY}(r)$ of the BY energy $E_{BY}(r)$ to the interior of a horizon proposed in [6]. Writing the Schwarzschild-like metric as
$$ds^2 = -\epsilon f(r)^2 dt^2 + \epsilon f(r)^{-2} dr^2 + r^2 d\Omega^2 \ , \tag{21}$$
with $\epsilon = \pm 1$ corresponding to the exterior (interior) region, the definition of [6] is
$$E_{LSY}(r) = \frac{r}{G_N}(1 - \epsilon f(r)) \tag{22}$$
($E_{LSY}(r) = E_{BY}(r)$ in the region $\epsilon = +1$). We now write the effective potential equation for radial timelike ($\lambda = +1$) or spacelike ($\lambda = -1$) geodesics as
$$\tfrac{1}{2}\dot{r}^2 + V_{\text{eff}}^{\lambda}(r) = E_{\text{eff}}^{\lambda} \tag{23}$$
where $E_{\text{eff}}^{\lambda} = \frac{1}{2}(E^2 - \lambda)$. Then one easily finds
$$\epsilon\lambda V_{\text{eff}}^{\lambda}(r) = \tfrac{1}{2}\left((1 + V_{LSY}(r))^2 - \epsilon\right) \ . \tag{24}$$
This relation between the effective potential $V_{\text{eff}}^{\lambda}(r)$ and $1 + V_{LSY}(r)$ is identical to the relation between the effective Newtonian energy E_{eff}^{λ} and the relativistic geodesic energy E for any ϵ provided that one correlates the region of interest (specified by ϵ) with the character of the geodesic (indicated by λ) by making the choice $\epsilon = \lambda$. Thus for $\epsilon = -1$ $E_{LSY}(r)$ appears to be naturally associated with spacelike geodesics.

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